An introduction to variational methods

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Damien Galant

CERAMATHS/DMATHS
Université Polytechnique Hauts-de-France

Département de Mathématique
Université de Mons
F.R.S.-FNRS Research Fellow

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1 Introduction

2 Vibrating plates, eigenfunctions of the Dirichlet Laplacian and their nodal sets

3 A few phenomena on metric graphs

An example of a "variational result": Rolle's Theorem

Theorem (Rolle)

Let $a, b \in \mathbb{R}$ be so that $a < b$. If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, differentiable on $]a, b[$ and such that $f(a) = f(b)$, then there exists $\xi \in \,]a, b[$ such that $f'(\xi) = 0$.

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Proof.

On the blackboard! □

Main message

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- We convert this question into the search for *minimizers/maximizers* of a certain function, namely f.
- When looking for extremizers, we can use *compactness arguments*.

Optimizing under constraints, Lagrange multipliers

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Theorem (Lagrange's multiplier Theorem with one constraint)

Let $f, g : U \to \mathbb{R}$ be real valued functions defined on some open set U. Then, if $a \in U$ is so that $g(a) = 0$ and that a minimizes locally $f(x)$ under the constraint $g(x) = 0$, then

$$\text{Either } \nabla g(a) = 0 \text{ or there exists } \lambda \in \mathbb{R} \text{ such that } \nabla f(a) = \lambda \nabla g(a).$$

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Example

$U = \mathbb{R}^2$; $f(x, y) := y$; $g(x, y) := x^2 + y^2 - 1$. On the blackboard!

An application: spectral theory of symmetric matrices

Theorem

Let $A \in \mathbb{R}^{N \times N}$ be a symmetric real matrix. Then, there exists a sequence

$$\lambda_1 \leq \cdots \leq \lambda_N$$

of real eigenvalues and a sequence of orthnormal eigenvectors $(\varphi_1, \ldots, \varphi_N) \in \left(\mathbb{R}^N\right)^N$ so that

$$A\varphi_i = \lambda_i \varphi_i$$

for every $1 \leq i \leq N$.

An application: spectral theory of symmetric matrices

Proof.

Let us define a *quadratic form* $q : \mathbb{R}^N \to \mathbb{R}$ by

$$q(u) := (u \mid Au),$$

where $(\cdot \mid \cdot)$ is the usual scalar product on $\mathbb{R}^N$: $(u \mid v) := \sum_{1 \le i \le N} u_i v_i$.

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Since the unit sphere of $\mathbb{R}^N$ is compact, there exists $\varphi_1 \in \mathbb{R}^N$ so that $\|\varphi_1\| = 1$ and that

$$q(\varphi_1) = \min_{\|u\|=1} q(u).$$

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Therefore, the gradient of q is proportional to the gradient of $u \mapsto \|u\|^2$, namely proportional to u.

An application: spectral theory of symmetric matrices

Proof (continued).

The gradient of q is given by

$$\nabla q(u) = Au.$$

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Proof (continued).

We now decompose $\mathbb{R}^N$ as

$$\mathbb{R}^N = \mathbb{R}\varphi_1 \oplus \varphi_1^\perp,$$

where

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We remark that A maps $\varphi_1^\perp$ to itself. Indeed, for any $u \in \varphi_1^\perp$, we have that

$$(Au \mid \varphi_1) = (u \mid A\varphi_1) = \lambda_1(u \mid \varphi_1) = 0.$$

An application: spectral theory of symmetric matrices

Proof (continued).

Therefore, one may repeat the same argument as above to the function

$$q_{|\varphi_1^\perp} : \varphi_1^\perp \to \mathbb{R} : u \mapsto (u \mid Au).$$

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We thus get the existence of $(\lambda_2, \varphi_2) \in \mathbb{R} \times \mathbb{R}^N$ so that

$$A\varphi_2 = \lambda_2 \varphi_2, \qquad \varphi_2 \perp \varphi_1.$$

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We now write

$$\mathbb{R}^N = \mathbb{R}\varphi_1 \oplus \mathbb{R}\varphi_2 \oplus \langle \varphi_1, \varphi_2 \rangle^\perp$$

and iterate the minimization argument, which ends the proof. □

Spectral theory of symmetric matrices: a summary

Theorem

The eigenvalues $\lambda_1 \leq \cdots \leq \lambda_N$ of a symmetric matrix A are given by

$$\lambda_i = \min_{\substack{\|u\|=1 \\ u \perp \varphi_1 \\ \vdots \\ u \perp \varphi_{i-1}}} (u \mid Au),$$

where $\varphi_1, \dots, \varphi_N$ are the associated eigenvectors.

The Min-max Theorem

Theorem

There exists a sequence

$$\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_N$$

of eigenvalues, with a sequence of orthnormal eigenvectors $\varphi_1, \varphi_2, \ldots, \varphi_N$*. Moreover, the* k*th eigenvalue is given by*

$$\lambda_k = \inf_{\substack{V \subseteq \mathbb{R}^N \\ \dim V = k}} \sup_{\substack{u \in V \\ \|u\|=1}} (u, Au).$$

Chladni figures

Source: https://www.youtube.com/watch?v=wvJAgrUBF4w

Modeling vibrating plates

Vibrations of a plate of shape $\Omega \subset \mathbb{R}^2$ are described by the wave equation

$$\begin{cases} \partial_{tt} u(t,x) = \Delta u(t,x), & (t,x) \in [0,+\infty[\times \Omega, \\ \\ \end{cases}$$

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The spectral problem

We consider a bounded open set $\Omega \subset \mathbb{R}^N$, with a regular boundary (say that $\partial\Omega$ is a $\mathcal{C}^\infty$ submanifold of $\mathbb{R}^N$).

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Using eigenfunctions in the initial value problem

If u_0 is an eigenfunction with eigenvalue λ, then the associated solution of the wave equation is given by

$$u(t,x) = \cos(\sqrt{\lambda}t)u_0(x).$$

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This solution:

- is periodic in time;
- if $u_0(x) = 0$, then $u(t,x) = 0$ for all t.

The case of dimension one: spectrum

Eigenvalue problem: find $(\lambda, u) \in \mathbb{R} \times \mathcal{C}^2(0, L)$ so that

$$\begin{cases} -u''(x) = \lambda u(x), \quad x \in (0, L), \\ u(0) = u(L) = 0. \end{cases}$$

Example

Computations on the blackboard!

The case of dimension one: wave equation

Source: `https://www.youtube.com/watch?v=QxEP6LINeR8`

Another example: the square in $\mathbb{R}^2$

Eigenvalue problem: find $(\lambda, u) \in \mathbb{R} \times \mathcal{C}^2((0,L)^2)$ so that

$$\begin{cases} -\Delta u(x,y) = \lambda u(x,y), & (x,y) \in (0,1)^2, \\ u(x,0) = u(x,L) = 0, & x \in (0,1), \\ u(0,y) = u(L,y) = 0, & y \in (0,1). \end{cases}$$

Example

Computations on the blackboard!

Nodal sets of eigenfunctions of the square

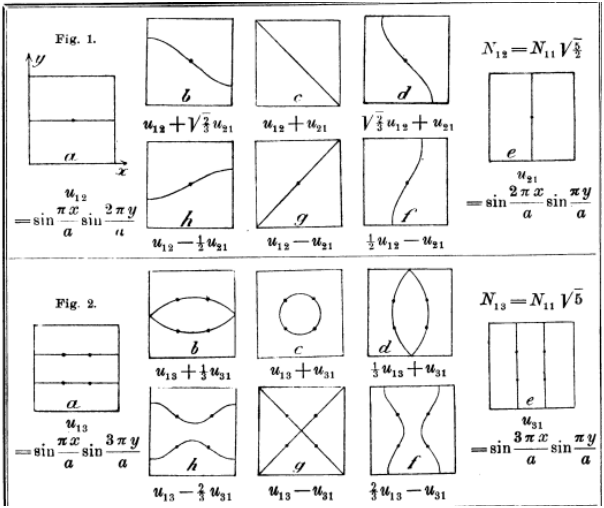

Figure: Above: $\lambda_2 = \lambda_3 = \pi^2(1^2+2^2)/L^2$; below: $\lambda_5 = \lambda_6 = \pi^2(1^2+3^2)/L^2$.

Nodal sets of eigenfunctions of the square

The previous image was taken from

F. Pockels, Über die partielle Differentialgleichung $\Delta u + k^2 u = 0$ und deren Auftreten in mathematischen Physik, Teubner-Leipzig, 1891, Historical Math. Monographs. Cornell University http://ebooks.library.cornell.edu/cgi/t/text/text-idx?c=math;idno=00880001.

Qualitative properties of the first eigenfunction

Theorem

The infimum

$$\inf_{\|u\|_{L^2(\Omega)}=1} \int_\Omega |\nabla u|^2.$$

is attained by the a function φ_1*. This function is* $\mathcal{C}^2(\overline{\Omega})$*, solves*

$$\begin{cases} -\Delta\varphi_1(x) = \lambda_1\varphi_1(x), & x \in \Omega, \\ \varphi_1(x) = 0, & x \in \partial\Omega, \end{cases}$$

and one has that $\varphi_1(x) > 0$ *for all* $x \in \Omega$*.*

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and one has that $\varphi_1(x) > 0$ *for all* $x \in \Omega$.

The positivity result follows from the maximum principle for the Laplacian.

Courant–Fischer Min-max Theorem

Theorem

There exists a sequence

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$$

of eigenvalues of the Laplace operator $-\Delta$ *with Dirichlet boundary conditions, with a sequence of eigenfunctions* $\varphi_1, \varphi_2, \cdots$ *which is orthonormal in* $L^2(\Omega)$*. Moreover, the* k*th eigenvalue is given by*

$$\lambda_k = \inf_{\substack{V \subseteq H_0^1(\Omega) \\ \dim V = k}} \sup_{\substack{u \in V \\ \|u\|_{L^2(\Omega)} = 1}} \|\nabla u\|_{L^2(\Omega)}^2.$$

Monotonicity of eigenvalues

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Main message

Smaller domains have larger eigenvalues!

Courant's Theorem

Definition (Nodal domain)

A *nodal domain* of a function $u : \Omega \to \mathbb{R}$ is defined as a connected component of

$$\left\{x \in \Omega \mid u(x) \neq 0\right\}.$$

Theorem (R. Courant (1923))

An eigenfunction associated with the kth eigenvalue has at most k nodal domains.

Courant's Theorem: sketch of proof

Sketch of proof following Bérard and Helffer (see references).

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Let u be an eigenfunction associated with λ_k.
Assume that u has at least $k+1$ nodal domains, say $\omega_1, \omega_2, \dots$. For any $1 \leq j \leq k$, we define

$$u_j(x) := \begin{cases} u(x) & \text{if } x \in \omega_j \\ 0 & \text{otherwise.} \end{cases}$$

Courant's Theorem: sketch of proof

Proof.

One can find a linear combination

$$v := \sum_{1 \leq j \leq k} \alpha_j u_j$$

such that v is orthogonal to $\varphi_1, \ldots, \varphi_{k-1}$ and one has $\|v\|_{L^2(\Omega)} = 1$.

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Proof.

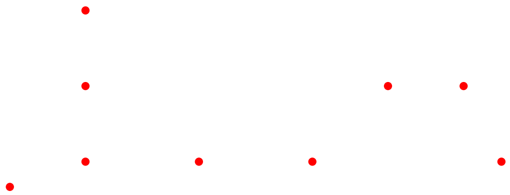
From the definition of u_j, it follows that

$$\int_\Omega |\nabla v|^2 \, \mathrm{d}x = \lambda_k.$$

Therefore, using the Min-max principle, v is also an eigenfunction associated with λ_k. However, *using the unique continuation principle*, v vanishes identically, since it vanishes on some open set. This contradicts the fact that $\|v\|_{L^2(\Omega)} = 1$. □

What is a compact metric graph?

A compact metric graph is made of a finite number of vertices



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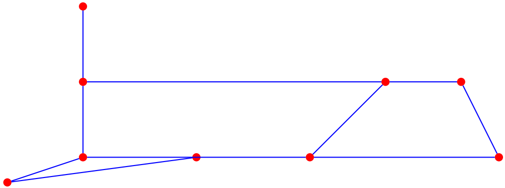

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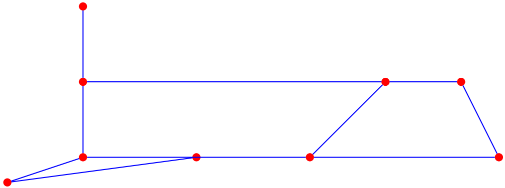
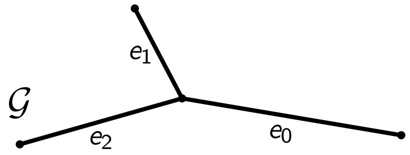


Metric graphs: the length of edges are important.

Functions defined on metric graphs



A compact metric graph $\mathcal{G}$ with three edges e_0 (length 5), e_1 (length 4) and e_2 (length 3)

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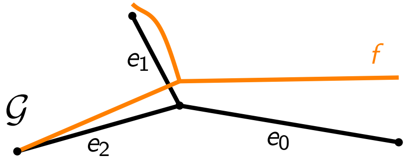

A compact metric graph $\mathcal{G}$ with three edges e_0 (length 5), e_1 (length 4) and e_2 (length 3), a function $f : \mathcal{G} \to \mathbb{R}$

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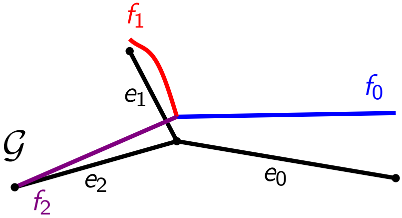

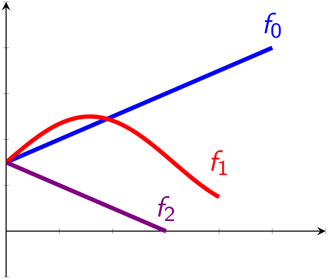

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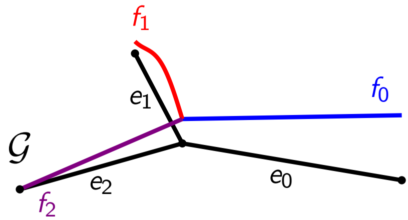

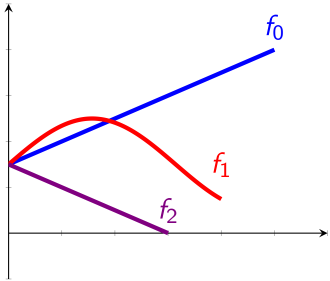

A compact metric graph $\mathcal{G}$ with three edges e_0 (length 5), e_1 (length 4) and e_2 (length 3), a function $f : \mathcal{G} \to \mathbb{R}$, and the three associated real functions.

$$\int_{\mathcal{G}} f \,\mathrm{d}x \overset{\text{def}}{=} \int_0^5 f_0(x)\,\mathrm{d}x + \int_0^4 f_1(x)\,\mathrm{d}x + \int_0^3 f_2(x)\,\mathrm{d}x$$

The spectral problem on metric graphs

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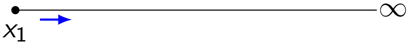
where the symbol $e \succ \mathrm{v}$ means that the sum ranges over all edges of vertex v and where $\frac{\mathrm{d}u}{\mathrm{d}x_e}(\mathrm{v})$ is the outgoing derivative of u at v (*Kirchhoff's condition*).

Kirchoff's condition: degree one nodes



$$\lim_{\substack{t \to 0 \\ t>0}} \frac{u(x_1 + t) - u(x_1)}{t} = 0$$

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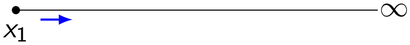
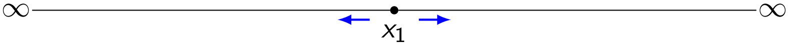


$$\lim_{\substack{t \to 0 \\ t>0}} \frac{u(x_1+t)-u(x_1)}{t} = 0$$

In other words, the derivative of u at x_1 vanishes: this is the usual Neumann condition.

Kirchoff's condition: degree two nodes



$$\left(\lim_{\substack{t \to 0 \\ t>0}} \frac{u(x_1+t)-u(x_1)}{t}\right) + \left(\lim_{\substack{t \to 0 \\ t>0}} \frac{u(x_1-t)-u(x_1)}{t}\right) = 0$$

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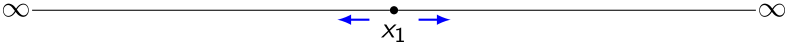

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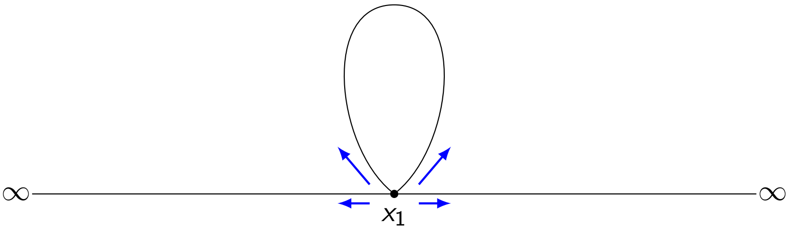
In other words, the left and right derivatives of u are equal, which simply means that u is differentiable at x_1. This explains why usually we do not put degree two nodes.

Kirchoff's condition in general: outgoing derivatives



$$\sum_{e \succ \mathrm{v}} \frac{\mathrm{d}u}{\mathrm{d}x_e}(\mathrm{v}) = 0$$

The Sobolev space $H^1_Z(\mathcal{G})$

We work on the Sobolev space

$$H^1_Z(\mathcal{G}) := \left\{ u : \mathcal{G} \to \mathbb{R} \mid u \text{ is continuous; } u(\mathrm{v}) = 0 \text{ for all } v \in Z, u' \in L^2(\mathcal{G}) \right\}.$$

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When applying the Min-max method, we will obtain a couple $(\lambda, \varphi) \in \mathbb{R} \times H^1_Z(\mathcal{G})$ so that

$$\int_{\mathcal{G}} \varphi' \psi' \, \mathrm{d}x = \lambda \int_{\mathcal{G}} \varphi \psi \, \mathrm{d}x$$

for all $\psi \in H^1_Z(\mathcal{G})$.

Recovering the equation

If ψ has compact support in the interior of an edge $e = \mathrm{AB}$, we have

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so that $-\varphi'' = \lambda\varphi$ on edges of $\mathcal{G}$.

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&= \sum_{1 \le i \le D} \Big(\frac{\mathrm{d}u}{\mathrm{d}x_{e_i}}(\mathrm{B}_i) \underbrace{\psi(\mathrm{B}_i)}_{=0} - \frac{\mathrm{d}u}{\mathrm{d}x_{e_i}}(\mathrm{A}_i) \underbrace{\psi(\mathrm{A})}_{=1} \Big) \\
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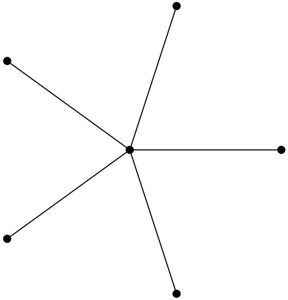
so that $\sum_{1 \le i \le D} \frac{\mathrm{d}\varphi}{\mathrm{d}x_{e_i}}(\mathrm{A}_i) = 0$, which is Kirchhoff's condition.

A surprising phenomena: compact star graphs with Dirichlet conditions



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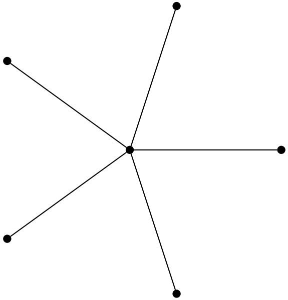

Example

Computations on the blackboard!

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- At the end of the proof of Courant's Theorem, we used *unique continuation principles*.
- Such unique continuation principles do *not* hold in the metric graph setting, as shown by the eigenfunctions vanishing identically on edges.
- Solutions to *nonlinear* problems on metric graphs may also exhibit this phenomena of being identically zero on some edges (see the arXiV preprint in the references.)

Thanks for your attention!

Curious about metric graphs?

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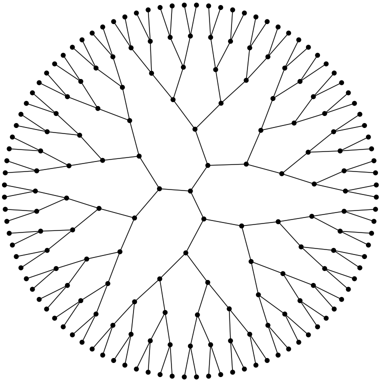
NQG : Summer school : "Nonlinear Quantum Graphs"



17–21 June 2024, Valenciennes; https://nqg.sciencesconf.org/

To go further: spectral problems

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As for matrices, we want to show that

$$\lambda_1 = \min_{\|u\|_{L^2(\Omega)}=1} \int_\Omega |\nabla u|^2.$$

Some care is needed

- To find a minimizer, we need some compactness. However, there is often a lack of compactness when working in functional spaces (if E is a normed vector space, then $B[0,1]$ is compact if and only if $\dim E < \infty$);

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- Weak convergence is indeed weaker than strong convergence: if $\dim H = +\infty$ is separable and $(e_n)_n$ is an Hilbert basis, then

$$e_n \underset{n\to\infty}{\rightharpoonup} 0.$$

The Sobolev space $H_0^1(\Omega)$

As apparent in the previous discussion, we would like to use

$$(u \mid v)_{H^1} := \int_\Omega \nabla u \cdot \nabla v + \int_\Omega u\, v$$

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Remark: H^1_0: we start from $\mathcal{C}^\infty_c(\Omega)$, so the functions are equal to 0 on $\partial\Omega$.

A few properties in the space $H_0^1(\Omega)$

Distributional derivatives

- The space $H_0^1(\Omega)$ is the space of $L^2(\Omega)$ functions which admit a *distributional gradient* $\nabla u \in \left(L^2(\Omega)\right)^N$ and which vanish on $\partial\Omega$.

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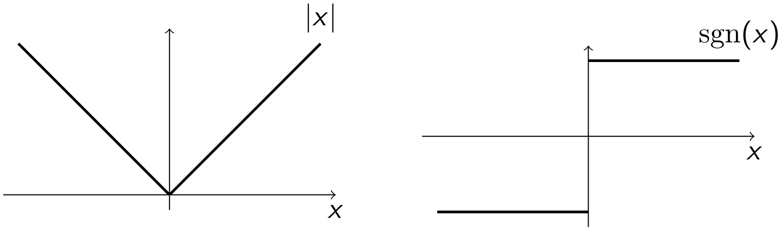
- Compatibility with the absolute value: if $u \in H_0^1(\Omega)$, then $|u|$ belongs to $H_0^1(\Omega)$ and ∇u and $\nabla|u|$ have the same norm.

An example of the weak derivative: the absolute value and the sign function

The function $\mathbb{R} \to \mathbb{R} : x \mapsto |x|$ has a weak derivative given by $x \mapsto \operatorname{sgn}(x)$.



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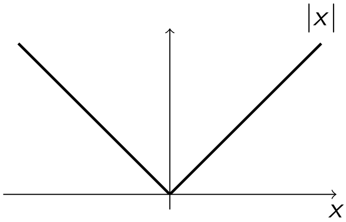

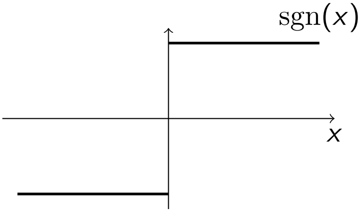

The sign function does not have a weak derivative on $\mathbb{R}$.

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$$u_n \xrightarrow[n\to\infty]{L^q(\Omega)} u,$$

for all $2 \leq q \leq 2^*$, where

$$2^* := \begin{cases} \infty & \text{for } N = 1 \text{ and } N = 2, \\ \frac{2N}{N-2} & \text{otherwise.} \end{cases}$$

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- Weak lower semicontinuity: if $(u_n)_n \subseteq H_0^1(\Omega)$ converges weakly to $u \in H_0^1(\Omega)$, then

$$\|\nabla u\|_{L^2(\Omega)} \leq \liminf_n \|\nabla u_n\|_{L^2(\Omega)}.$$

Existence of the first eigenfunction

Proof.

Let $(u_n)_n \subseteq H^1_0(\Omega)$ be a minimizing sequence for the problem. One has $\|u_n\|_{L^2} = 1$ for every n.

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Thus, u is the required minimizer. □

Existence of the second eigenfunction

Theorem

The infimum

$$\inf_{\substack{u \in H_0^1(\Omega) \\ \|u\|_{L^2(\Omega)}=1 \\ (\nabla u|\nabla \varphi_1)_{L^2}=0}} \int_\Omega |\nabla u|^2$$

is attained by a $H_0^1(\Omega)$ *function.*

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$$0 = (\nabla u_n \mid \nabla \varphi_1)_{L^2} \xrightarrow[n \to \infty]{} (\nabla \varphi_2 \mid \nabla \varphi_1)_{L^2}$$

by weak convergence, so that $(\nabla \varphi_2 \mid \nabla \varphi_1)_{L^2} = 0$. □